Energy principle application to response of viscoelastic bars

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Research Paper - Subject review

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A wide range of practical engineering problems exists for which obtaining exact solutions directly is challenging. This is because of the complex nature of the governing differential equations or the difficulties arising from the boundary and initial conditions of the problem. To address these problems, scalar quantities, such as work and energy, are used as an alternative approach. The virtual work principle constitutes the basis for the energy and variational formulations. This study uses energy concepts to formulate viscoelastic structures and discuss the statically indeterminate axially loaded viscoelastic bar problem. A simple and efficient energy-based formulation for analysis is proposed. The total potential energy (TPE) expression in terms of the displacements of the nodes was obtained in Laplace space. The solutions that minimise the TPE expression are real displacements, and the inverse Laplace transform method is applied to transform the function back into the time domain. Different examples were considered to ensure accuracy and demonstrate the potential of the proposed solution technique. This method is convenient for obtaining a solution directly by following a few simple process steps, regardless of the change in the viscoelastic material model, the number of elements in the system, and the type of loading.

Key words:

viscoelastic, time-dependent analysis, potential energy principle, statically indeterminate, Laplace domain, inverse Laplace

Pregledni rad

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Primjena načela energije u izračunavanju odziva viskoelastičnih štapova

Za široki je raspon praktičnih problema u građevinarstvu teško pronaći točna rješenja. Razlog je tome u složenosti diferencijalnih jednadžbi koje opisuju problem ili u poteškoćama koje proizlaze iz rubnih i početnih uvjeta. Kao alternativni pristup rješavanju tih problema primjenjuju se skalarne veličine poput rada i energije. Načelo virtualnoga rada osnova je energijskih i varijacijskih pristupa. U ovom je istraživanju pomoću energijske formulacije viskoelastičnih konstrukcija obrađen problem statički neodređenoga viskoelastičnog aksijalno opterećenoga štapa. Za analizu je predložena jednostavna i učinkovita formulacija utemeljena na energiji. Izraz za ukupnu potencijalnu energiju u ovisnosti o pomacima čvorova izveden je u Laplaceovom prostoru. Rješenja koja smanjuju ukupnu potencijalnu energiju stvarni su pomaci, a inverzna Laplaceova transformacija primijenjena je za preslikavanje funkcije nazad u vremensku domenu. Analizirani su različiti primjeri kako bi se osigurala točnost i prikazao potencijal predložene metode. Ovom se metodom rješenje može dobiti neposredno u nekoliko jednostavnih koraka, neovisno o promjeni modela viskoelastičnoga materijala, broju elemenata u sistemu i vrsti opterećenja.

Ključne riječi:

viskoelastičnost, analiza ovisna o vremenu, načelo potencijalne energije, statički neodređen, Laplaceova domena, inverzna Laplaceova transformacija

1. Introduction

Linear elastic material behaviour is the simplest and most common idealisation. In most materials, such as polymers, metals at high temperatures, wood, and living tissue and cells, the effects of both elasticity and viscosity are noticeable. These effects are called viscoelasticity (VE). Instantaneous elasticity, creep, stress relaxation, instantaneous elastic recovery, delayed recovery, and permanent or viscous deformations occur in VE bodies. The deformation of VE materials is affected by the loading rate and duration of the applied load. Proper modelling of the time-dependent behaviour is essential for the accurate analysis of the structures made of VE materials. The behaviour of the VE materials was mathematically modelled using mechanical analogues consisting of springs and dashpots. The spring and dashpot are the basic elements used to model the elastic and viscous behaviours of VE materials, respectively. Various combinations of spring and dashpot elements in series and/or parallel have been used to describe a wide range of linear VE materials.

Different classification methods are used for engineering structures. The most common classification is based on a combination of the geometric configuration and loading characteristics of the structure. A bar (or rod) is the simplest structure from a geometric perspective. Depending on whether the load is tension, compression, torsion, or bending, a straight bar is often referred to as a rod, column, shaft, or beam.

Linear VE has been used for structural analysis of time-dependent materials. Laplace transformation, Fourier transformation, and the correspondence principle (the elastic–viscoelastic analogy) are used for analysing viscoelastic problems. The Laplace transform was applied to VE beams in Ref. [1]. A previous study used the elastic–viscoelastic correspondence principle [2] to solve the VE beam field equations. Christensen [3] analysed the viscoelastic beam problem using the Fourier transform technique. Several studies have been conducted on the static and dynamic analyses of VE beam-type structures. The quasi-static and dynamic (QS-DYN) behaviour of the linear VE and conventional Timoshenko beam was analysed in Ref. [4] using the Prony series. The time terms were removed using a hybrid Laplace transform. Based on Hamilton's principle, a finite element (FE) solution can be obtained without performing an integral transformation. A previous study [5] reports the QS-DYN analysis of the VE Timoshenko beam using the hybrid Laplace–Carson and FE method. The study used a hereditary integral representation of the constitutive relations. In the numerical examples, Kelvin and three-parameter solid models are used, and for numerical Laplace inversion, the Fourier series and Schapery methods are considered.

Among the computational methods used for VE, the FE method is the most common and versatile. Several authors reported the application of the FE method to viscoelastic beam problems. [6] presented a procedure to determine the transient response of a VE-beam system. The multiple degrees of freedom (MDOF) FE model was used for the analysis. A fast Fourier transform (FFT) was applied to transform the frequency-domain equation into the time domain. [7] studied the VE-beam FE models based on different kinematic assumptions. Hamilton's principle was used to derive the governing equations. The results are transformed into the time domain using a numerical inversion procedure. [8] proposed a Golla–Hughes method (GHM)-based sandwich FE model. The proposed formulation is based on a second-order time-domain realisation of the Laplace-domain motion equations. The parameters characterising the viscoelastic material were experimentally determined. [9] developed a solid-shell FE method for free vibration analysis of VE sandwich beams. The general formulation of the problem is presented using different finite element discretisation while underlying their respective limitations. [10] developed an FE model based on the first-order shear deformation theory and the Hamilton principle for the vibration analysis of a sandwich beam with a viscoelastic material core sandwiched between two elastic layers. In the derivation of the FE equation, it was assumed that there was no relative sliding between the layers, and the effect of the moment of inertia was not considered. The numerical results were validated through the experimental testing of a VE sandwich cantilever beam. [11] applied analytical and numerical models to predict the serviceability limit states of flexural reinforced-concrete members. The considered beam elements were investigated under both short- and long-term bending loads. Two distinct mathematical models were applied in the analysis, namely a model with beam finite elements (1D analysis) and a model with plane stress elements (2D analysis). For long-term analysis, the time-dependent effects of concrete, creep, and shrinkage were employed in both models through the viscoelastic concrete behaviour. Numerical and analytical models were validated using existing experimental data. Most studies available in the literature provide good coverage of the vibration characteristics and stability of VE beams [12-25].

This study aims to apply the total potential energy (TPE) principle to determine the VE behaviour of bar-type structures. The TPE principle has a wide range of applications and is particularly adaptable to engineering structures, which are classified based on a combination of geometric configurations and loading characteristics, such as beams, frames, plates, and shells. However, these structures are assumed to be elastic (i.e., the magnitude of the response quantity, such as stress and displacement, is proportional to the magnitude of the applied load).

Initially, researchers usually focused on elastic problems; however, when viscoelastic materials are widely utilised in practice, the need for studies on viscoelastic problems has increased. Assuming that the material is elastic, the

simplification of the analysis proves to be inconsistent with reality because most engineering materials exhibit noticeable time effects and are viscoelastic because of internal friction. Therefore, viscoelastic constitutive relations are more realistic than elastic constitutive relations in reflecting material behaviour. Consideration of the time-dependent feature of the VE material and the formation of the basis for real structural analysis has drawn much attention in research. The author and co-workers published studies devoted to implementing a simple and efficient computing procedure for the QS-DYN analyses of VE structural elements based on the concept of the Gâteaux differential. New functionals suitable for mixed FE formulations in the Laplace-Carson domain were developed for VE Kirchhoff plates by [26] and first-order shear deformable Mindlin–Reissner plates by [27].

The motivation for this study was derived from a literature survey. The principles of work and energy are fundamental and form the basis of various approximate methods of analysis. Despite extensive research on the development and application of work and energy methods for the analysis of elastic structural elements, this study proposes a total-energy principle for viscoelastic problems. It is well known that there exists no explicit total energy representation for viscous problems in the time domain because the viscous term plays a non-conservation role in a structural system. In general, the virtual work principle is established for viscoelastic problems in time-domain computation. In the present study, it was demonstrated that an explicit total energy representation for viscous problems can exist in the Laplace domain.

In certain cases, viscoelastic evaluation is sufficiently complicated because modelling the behaviour of progressive deformations (i.e. suffering from creep, relaxation, and hysteresis problems) along the long-term structural response involves mathematical complexity. The main objective of this study is to discuss the linear VE behaviour of bar-type structures using the principles of work and energy. This is the first study that presents a new, simple, and efficient energy-based formulation for the analysis of statically indeterminate VE bar-type structures. The proposed formulation is theoretically simple and easy to implement in practice. This proved to be efficient for obtaining the results of any constitutive model of linear viscoelasticity. Rebuilding the set of equations to be solved depending on the types of VE material models, loadings, geometries, and number of elements in which the body is divided into a few straightforward steps is the primary advantage of the proposed method. This study presents highly accurate solutions for both statically determinate and indeterminate VE systems. In the FE analysis, it is essential to use fine meshing to obtain more precise results; however, fine meshing is often time-consuming. The proposed method yields satisfactory results independent of the number of subdomains (elements) in which the body is divided, thereby saving time. The Kelvin-Voight model and standard linear solid (SLS) model were used to develop the

VE constitutive equations. Using the convolution properties of the Laplace transform, the minimum potential energy principle was applied to analyse an axially loaded statically indeterminate bar-type structure. Because the governing equations are time-dependent, a solution method is proposed. Several examples are considered to validate the proposed formulation. Beginning with a simple bar structure, this study takes the author and her co-workers through increasingly complex problems of solid mechanics, such as VE structures, which consist of an assemblage of two or more bars and are subjected to bending, shear, and axial loads, called frames, and subject to axial forces, only called truss in 2D or 3D, and problems involving deformations, stability, and vibrations of VE bodies. The methodology discussed in this study can be easily applied to the analysis of different types of VE structural members. It is simple, reliable, and efficient in terms of computation, and its results are accurate. The results of this study can serve as a benchmark for future research.

2. Methodology

There are two different approaches for the analysis of engineering structures. One method aims to describe the states of stress and deformation throughout the structure by integrating differential equations that satisfy the appropriate boundary conditions. This approach has the advantage of generality; however, performing integration analytically seems nearly always impossible. Another alternative approach is to use scalar quantities such as work and energy. The first law of thermodynamics requires that the sum of the mechanical energy W_E and heat energy Q supplied to the system be equal to the sum of the kinetic energy T and internal energy U as follows:

$$W_E + Q = T + U \tag{1}$$

Assuming that the deformation process is adiabatic, $Q = 0$. Additionally, the time at which the loads were applied very slowly (quasi-static) and a state of equilibrium was maintained; thus, $T = 0$. Under these assumptions, the first law is a restatement of energy conservation, such that

$$W_E = U \tag{2}$$

The work performed by the applied forces during the loading process (W_E) can be visualised as the area enclosed under the force curve. The work expression becomes:

$$W_E = \frac{1}{2}\int_{x_1}^{x_2} F(x)dx \tag{3}$$

The applied force, denoted by F may represent a concentrated or distributed force or moment. Therefore, F and x are the generalised force and displacement, respectively.

The internal energy U (called the strain energy) must be derived. The work done by the external load is stored as strain (internal)

energy U. To determine the strain energy, the material behaviour must be defined; therefore, the strain energy can be expressed in the form of stress. The time dependence of the VE response was comparably more involved than those of the elastic response of solids and the viscous response of fluids. Hysteresis is observed in the stress-strain curve of a VE material, and the stiffness of the material depends on the strain rate $\dot{\varepsilon}$ = dε/dt or stress rate $\dot{\sigma}$ = dσ/dt. Based on the assumption that the material is a linear VE, it follows that the mechanical work done by the applied forces is regained, and the energy lost during the loading cycle is equal to the area of the loop, as shown in Figure 1.
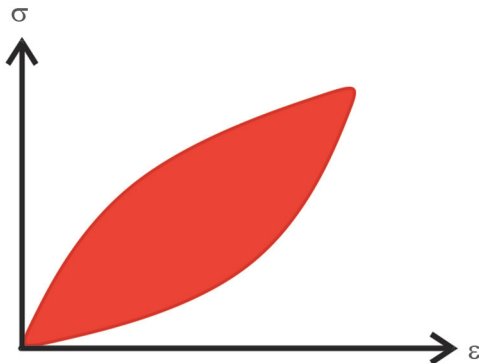

Figure 1. Stress-strain curve for a VE material

New forms of developed materials exhibit mechanical properties that are neither perfectly elastic nor show viscous fluid idealisation. Mechanical energy is stored in perfectly elastic materials without energy dissipation. Nevertheless, an ideal real (viscous) fluid can dissipate the entire energy input in a state of non-hydrostatic stress. VE materials can store and dissipate energy. The constitutive equations of time-dependent (VE) materials include stress, strain, and time variables. To define the constitutive equations, one form uses integrals, and the other uses differential equations.

The one-dimensional VE constitutive relationship of an integral form considering Boltzmann's superposition principle is expressed as follows:

$$\sigma(t) = M(t)\varepsilon(0^+) + \int_{0^+}^{t} M(t-\tau)\frac{d}{d\tau}\varepsilon(\tau)d\tau \tag{4}$$

$$\varepsilon(t) = C(t)\sigma(0^+) + \int_{0^+}^{t} C(t-\tau)\frac{d}{d\tau}\sigma(\tau)d\tau \tag{5}$$

where $M(t)$ is the relaxation modulus, and $C(t)$ is the creep compliance, both of which change with elapsed time t. Temperature dependence is omitted based on a uniform constant temperature assumption.

Eqns. (4-5) show that the material has memory; in other words, the material response is determined by the current value of the action in addition to all past actions.

The convolution integral allows the viscoelastic constitutive equations to be written in an elastic-like fashion [28] as follows:

$$\sigma(t) = M(t-\tau) * d\varepsilon(\tau) = M * d\varepsilon \tag{6}$$

$$\varepsilon(t) = C(t-\tau) * d\sigma(\tau) = M * d\sigma \tag{7}$$

As the constitutive equations of the linear VE material correspond to convolution integrals, the Laplace transform can be applied to convert them into algebraic equations. The Laplace transform for any Piecewise Continuous function $f(t)$ is defined as follows:

$$L(f(t)) = \bar{f}(s) = \int_0^\infty e^{-st}f(t)dt \tag{8}$$

Using the two properties of the Laplace transform (called the transforms of the derivatives in Eq. (A.1) and the convolution of Eq. (A.2); see Appendix), the constitutive equations result in the following algebraic form:

$$\bar{\sigma}(s) = s\bar{M}(s)\bar{\varepsilon}(s) \tag{9}$$

$$\bar{\varepsilon}(s) = s\bar{C}(s)\bar{\sigma}(s) \tag{10}$$

Here, s is the Laplace variable, and $\bar{\sigma}(s)$, $\bar{\varepsilon}(s)$, $\bar{M}(s)$ and $\bar{C}(s)$ are functions in the s-domain. Combining Eqns. (9) and (10) into an algebraic expression, the simple formula between the creep compliance and relaxation modulus in the Laplace transform plane is obtained as

$$\bar{M}(s)\bar{C}(s) = \frac{1}{s^2} \tag{11}$$

The expression for the strain energy per unit volume corresponds to the strain energy density; therefore, the expression is given by the integral on an interval (0, t), as follows [28]:

$$U_v = \frac{1}{2}\int_{\tau=0}^{t}\left[\int_{s=0}^{t} M(s-\tau)\frac{d\varepsilon(s)}{ds}ds\right]\frac{d\varepsilon(\tau)}{d\tau}d\tau \tag{12}$$

3. Total potential energy (TPE) principle

The application of variational principles to several branches of mechanics has a long history of development. The calculus of variations involves determining the function which makes a given functional stationary [29].

The principle of virtual work (PVW), also called the principle of virtual displacement (PVD), forms the basis of the variational formulation. It should be noted that the virtual work principle is applied independently of the constitutive relations of the material, whereas the stress-strain relations should be considered for the formulation of variational principles.

In this section, the TPE principle will be derived from PVW, such that

$$\delta W_E = \delta U \tag{13}$$

The PVW is equivalent to equilibrium equations for the system, and the external virtual work δW_E done by the real forces during the virtual distortion is equal to the internal virtual work δU done by the real stresses during the virtual distortion. Here, the symbol δ represents a variation operator.

Eqn. (13) may be written as follows regarding δ as the virtual operator:

$$\delta U - \delta W_E = \delta(U - W_E) = 0 \qquad (14)$$

In which,

$$(U - W_E) = \Pi \qquad (15)$$
and
$$\delta\Pi = 0 \qquad (16)$$

Here, Π is the TPE of the structure. The principle in Eqn. (16) indicates that among all displacements satisfying the prescribed boundary conditions, the actual displacement minimises the TPE.

4. Application of the TPE for the determination of time-dependent displacements in axially loaded statically indeterminate VE bar

The simplest application of the TPE principle is to determine the displacement of a statically indeterminate VE bar subjected to time-varying axial loading. One of the reasons for selecting this example is to illustrate the arguments underlying the method; another reason is that, to the authors' best knowledge, this is the first study devoted to the analysis of statically indeterminate members (axially loaded only) that show VE characteristics via a systematic procedure based on the energy principle.

In the case of a linear VE material, a constitutive relationship must be determined. VE behaviour can be modelled by considering simple mechanical analogues such as linear springs and dashpots. Simple rheological models widely used in theoretical studies consist of spring-dashpot systems, either in parallel or in series. The Maxwell model corresponds to a series arrangement and behaves as a fluid. The Kelvin–Voight model (Figure 2) corresponds to a parallel arrangement and exhibits solid behaviour.

The constitutive relationship for the Kelvin–Voight solid model is a linear first-order differential equation as follows:

$$\sigma = \sigma_S + \sigma_D = E\varepsilon + \eta\dot{\varepsilon} \qquad (17)$$

Eq. (17) contains the time derivatives. Therefore, stress and strain were not proportional. A Kelvin–Voight-based statically indeterminate VE bar with length L, fixed support at both ends, and loaded by two axial loads P_1 and P_2 (these axial loads can be a function of time or a constant) at $x = L_1$ and $x = L_1 + L_2$, respectively, is shown in Figure (3).
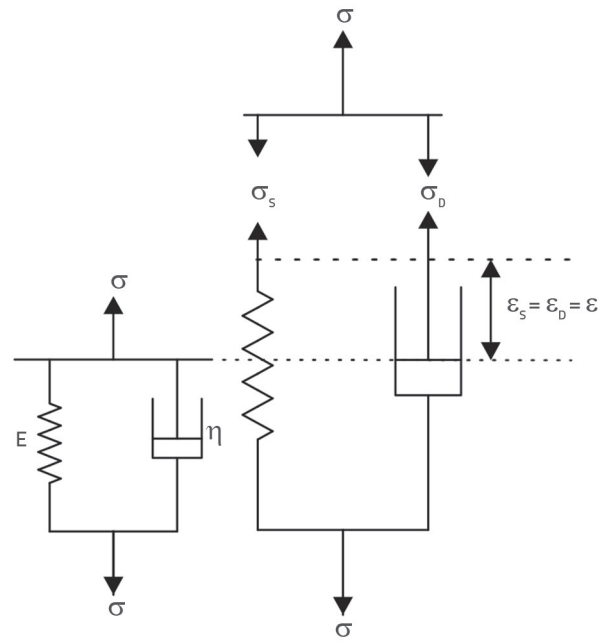

Figure 2. Schematic representation of Kelvin-Voight model and force balance

Our attention is restricted to the simpler case: the area of cross-section A is independent of x. Let us consider the bar to be composed of a set of $N = 3$ elements, each of length L_i ($i = 1$, N), as shown in Figure 3

The main aim was to construct a relationship between the nodal forces on an individual element and its end displacements. The time-dependent extension of the element is

$$\Delta_i = q_i(t) - q_{i-1}(t) \qquad (18)$$

The strain energy (U) of the bar is equal to the sum of the strain energies stored in all the members. In this case, the TPE in the Laplace domain is

$$\bar{\Pi} = \sum_{i=1}^{N} \bar{U}_i - \bar{W}_E \qquad (19)$$

where $\bar{W}_E$ is the Laplace transform of the external virtual work. By substituting the Laplace transform into Eqs. (17) into an internal energy expression rewritten in the Laplace domain and integrated over the entire volume of the structure yields the total strain energy $\sum_{i=1}^{N}\bar{U}_i$ in the Laplace domain:

To apply the TPE principle, the total potential ($\bar{\Pi}$) must be stated in terms of the unknown end displacements ($\bar{q}_1$ and $\bar{q}_2$) as shown in Figure 3.

$$\bar{\Pi} = \sum_{i=1}^{3} \bar{U}_i - \bar{P}_1\bar{q}_1 - \bar{P}_2\bar{q}_2 \qquad (20)$$

$\bar{\Pi}$ is the extremum of the true displacement field. Therefore, the variation in Eq. (20) for $\bar{q}_1$ and $\bar{q}_2$ yields the desired results in the Laplace domain.
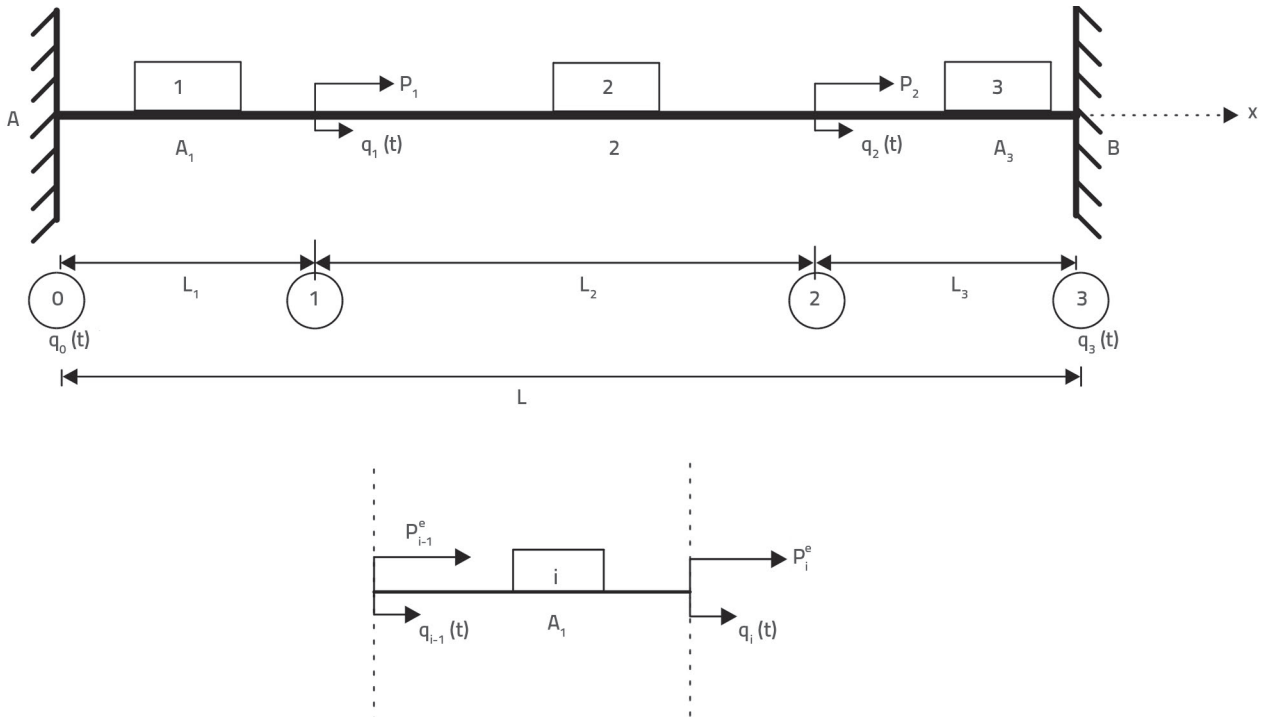
Figure 3. A schematic of the statically indeterminate VE bar

There are two independent quantities subject to variation in the functional: $\bar{q}_1$ and $\bar{q}_2$ with no subsidiary conditions. Taking variations with respect to these quantities requires $\delta\bar{\Pi} = 0$ which in the present case becomes

$$\frac{\partial\bar{\Pi}}{\partial\bar{q}_1} = 0$$
$$\frac{\partial\bar{\Pi}}{\partial\bar{q}_2} = 0$$

(21)

Eq. (21) provides a set of two simultaneous linear algebraic equations in which the two unknowns are $\bar{q}_1$ and $\bar{q}_2$. The matrix form of these equations is

$$EA\begin{pmatrix} \left(\dfrac{1}{L_1}+\dfrac{\eta s}{EL_1}+\dfrac{1}{L_2}+\dfrac{\eta s}{EL_2}\right) & \left(-\dfrac{1}{L_2}-\dfrac{\eta s}{EL_2}\right) \\ \left(-\dfrac{1}{L_2}-\dfrac{\eta s}{EL_2}\right) & \left(\dfrac{1}{L_2}+\dfrac{\eta s}{EL_2}+\dfrac{1}{L_3}+\dfrac{\eta s}{EL_3}\right) \end{pmatrix}\begin{Bmatrix} \bar{q}_1 \\ \bar{q}_2 \end{Bmatrix} = \begin{Bmatrix} \bar{P}_1 \\ \bar{P}_2 \end{Bmatrix}$$

(22)

where the matrices:

$$\left(\frac{EA_1}{L_1}+\frac{\eta sA_1}{L_1}\right)\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} \bar{q}_0 \\ \bar{q}_1 \end{Bmatrix} = \begin{Bmatrix} \bar{Q}_i^1 \\ \bar{Q}_j^1 \end{Bmatrix}$$

$$\left(\frac{EA_2}{L_2}+\frac{\eta sA_2}{L_2}\right)\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} \bar{q}_1 \\ \bar{q}_2 \end{Bmatrix} = \begin{Bmatrix} \bar{Q}_i^2 \\ \bar{Q}_j^2 \end{Bmatrix}$$

(23)

$$\left(\frac{EA_3}{L_3}+\frac{\eta sA_3}{L_3}\right)\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} \bar{q}_2 \\ \bar{q}_3 \end{Bmatrix} = \begin{Bmatrix} \bar{Q}_i^3 \\ \bar{Q}_j^3 \end{Bmatrix}$$

are known as element stiffness matrices. The properties of the individual elements were used to assemble a global stiffness matrix. Proceeding element-by-element leads to the global stiffness matrix, as in Eq. (24):

$$\begin{bmatrix} \left(\dfrac{EA_1}{L_1}+\dfrac{\eta sA_1}{L_1}\right) & \left(-\dfrac{EA_1}{L_1}-\dfrac{\eta sA_1}{L_1}\right) & 0 & 0 \\ \left(-\dfrac{EA_1}{L_1}-\dfrac{\eta sA_1}{L_1}\right) & \left(\dfrac{EA_1}{L_1}+\dfrac{\eta sA_1}{L_1}+\dfrac{EA_2}{L_2}+\dfrac{\eta sA_2}{L_2}\right) & \left(-\dfrac{EA_2}{L_2}-\dfrac{\eta sA_2}{L_2}\right) & 0 \\ 0 & \left(-\dfrac{EA_2}{L_2}-\dfrac{\eta sA_2}{L_2}\right) & \left(\dfrac{EA_2}{L_2}+\dfrac{\eta sA_2}{L_2}+\dfrac{EA_3}{L_3}+\dfrac{\eta sA_3}{L_3}\right) & \left(-\dfrac{EA_3}{L_3}-\dfrac{\eta sA_3}{L_3}\right) \\ 0 & 0 & \left(-\dfrac{EA_3}{L_3}-\dfrac{\eta sA_3}{L_3}\right) & \left(\dfrac{EA_3}{L_3}+\dfrac{\eta sA_3}{L_3}\right) \end{bmatrix}\begin{Bmatrix} \bar{q}_0 \\ \bar{q}_1 \\ \bar{q}_2 \\ \bar{q}_3 \end{Bmatrix} = \begin{Bmatrix} \bar{Q}_i^1 \\ \bar{Q}_j^1+\bar{Q}_i^2 \\ \bar{Q}_j^2+\bar{Q}_i^3 \\ \bar{Q}_j^3 \end{Bmatrix}$$

(24)

To solve this problem, the nodal force Q_i^e must be determined, and the equation system must be set up as follows:

$$F = Ku \qquad (25)$$

where F is the vector of the nodal forces for the entire structure, matrix K is the global stiffness matrix, and u is the nodal displacement vector. Condensed equations for the unknown displacements were created using the solutions of $\bar{q}_1$ and $\bar{q}_2$. The calculated displacements were used to obtain the $\bar{\Delta}_i$ Laplace domain extension values from the Laplace transforms of Eqs. (18). The inverse Laplace transform technique is used to convert the results obtained in the real (time) domain. Through the constitutive relation of the material, the normal forces can be easily calculated, if desired.

5. Numerical examples and discussion

In this section, three numerical applications are presented to ensure the accuracy and demonstrate the potential of the proposed procedure. For the first comparison example, the SLS VE material model was proposed based on a combination of springs and a dashpot to represent the VE behaviour of a statically determinate bar-type structure. Further examples were considered to present the solutions of statically indeterminate viscoelastic bar-type structures characterised by the Kelvin-Voight and SLS models. The steps for solving practical examples are as follows:
- Determine the constitutive relation of the viscoelastic material model in the Laplace domain
- Decide the number of sub domains that the body is divided
- The internal energy expression. Eq. (12), in the Laplace domain.
- By applying the TPE principle to the Laplace domain, see Eq. (20)
- Obtain unknowns (end-displacements) by using Eq. (24)
- Use the inverse Laplace transform to convert the results in the time domain.

To the best of our knowledge, no systematic energy-based formulations are available in the current literature. To verify the validity of the proposed TPE formulation, the results were compared with those of the ABAQUS finite element analysis because analytical solutions are not available in the literature. A constant Poisson's ratio ($\nu = 0.25$) was used for the analysis.

5.1. Comparison example: The SLS Model

This example, taken from [30], considers a rod with a length of 254 mm and diameter of 25.4 mm. One end of the rod was fixed, whereas the other end was subjected to suddenly applied axial stress with a magnitude of 0.689 MPa. The SLS model (see Figure 4) involves two systems in parallel. The Maxwell arm contains a spring ($E_2 = 62.01$ MPa) and a dashpot ($\eta = 62.01$ MPa.s), and the system contains only a spring ($E_1 = 6.89$ MPa) and is used for modelling the viscoelastic rod.
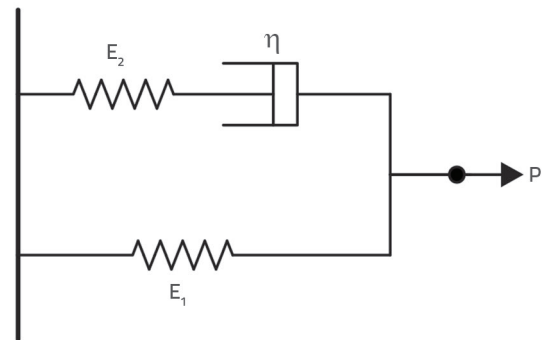

Figure 4. Rheological model of viscoelastic rod

The SLS material model has the following constitutive relation:

$$\sigma + \frac{\eta}{E_2}\dot{\sigma} = E_1\varepsilon + \frac{\eta(E_1 + E_2)}{E_2}\dot{\varepsilon} \qquad (26)$$

The strain-time history plot in the direction of the load obtained using the present TPE formulation and the SLS model in [30] is illustrated in Figure 5. The proposed and reference study solutions were in good agreement. Using this example, the computational performance and accuracy of the proposed energy-based formulation were tested for statically determinate systems.
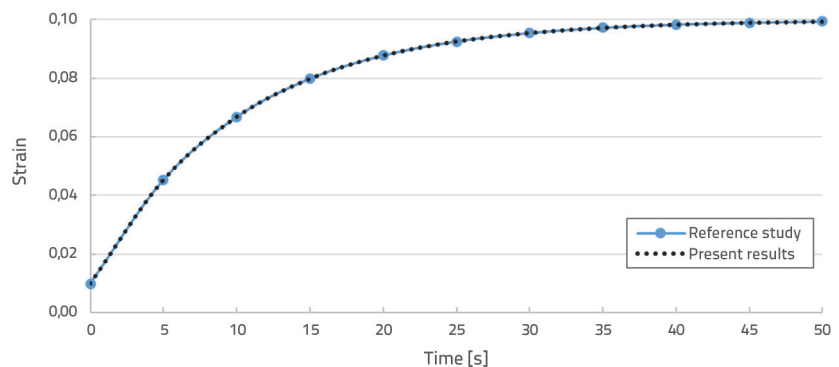

Figure 5. Comparison of solution of reference study and TPE formulation

5.2. Example 2: Statically indeterminate VE bar characterized by the Kelvin-Voight model

This example was used to evaluate the numerical accuracy of the TPE formulation. The statically indeterminate bar shown in Figure 6, which was fixed supported at both ends, was considered. The VE bar had a constant cross-sectional area A = 900 cm^2 and was subjected to loads of P_1 = 200 N and P_2 = 400 N. Let us determine the time-dependent behaviour specified by using the Kelvin-Voight VE material with E = 4×10^5 Pa and

η = 6×10^6 Pa·s. The accuracy of TPE application in an axially loaded VE bar with length $L = L_1+L_2+L_3$ comprising a set of N = 3 elements (L_1 = 1 m, L_2 = 2 m, and L_3 = 3 m), was validated against the results of an axially loaded elastic bar. To obtain the displacement of an axially loaded elastic bar in the direction of the load, the coefficient of viscosity η was set to near zero in the developed formulation. The elastic and VE results (see Figures (7) and (8)) correspond well, and it is again proven that the presented formulation is efficient for calculations.

5.3. Example 3: Statically indeterminate VE bar characterized by the SLS model

In this example, the time-dependent displacement values of the VE bar in Figure (6) characterised by the SLS model were calculated. The geometrical properties used in the previous example were considered in the analysis. The material properties of the VE bar are $E_1 = E_2$ = 4×10^5 Pa and η = 6×10^6 Pa·s. The system was subjected to axial forces with magnitudes P_1 = 200 N and P_2 = 400 N. The results were compared with those obtained using the ABAQUS Finite element analysis software. In ABAQUS, the bar was divided into 60 finite elements. The time-dependent displacements in the direction of the load are presented along with the ABAQUS results in Figure 9 and 10. The results of the TPE satisfactorily coincide with the corresponding ABAQUS results and are accurate to nearly five decimal places for different values of the time.

The mesh density is a critical issue in finite element analysis. Usually, a smaller mesh yields more accurate results; however, the computing time is also significant. The accuracy of the proposed energy-based formulation appeared to be independent of the number of elements in the system.

Figures 7-8 and Figures 8-9 depict the displacement versus time results for the Kelvin-Voight model and SLS models, respectively. As predicted, the spring of the Kelvin model wants to stretch, but is restrained by the dashpot, which cannot react immediately. Therefore, the spring length did not change when a load was suddenly applied to the Kelvin model.
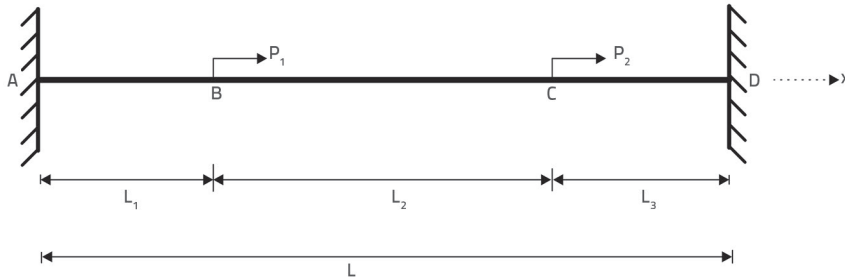

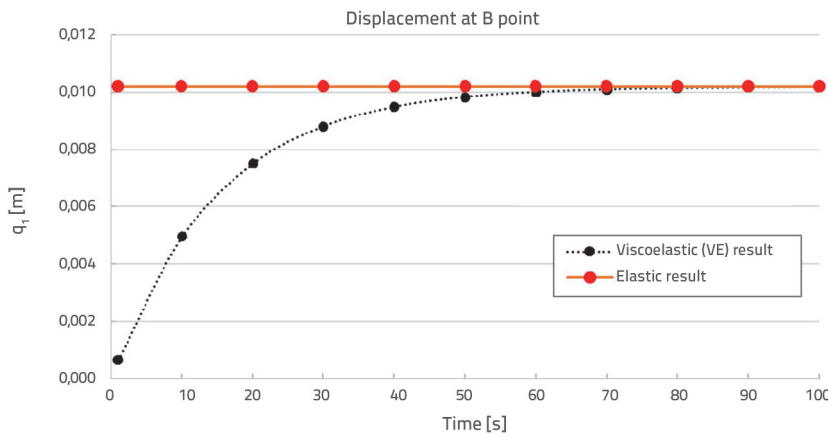Figure 6. Statically indeterminate VE bar

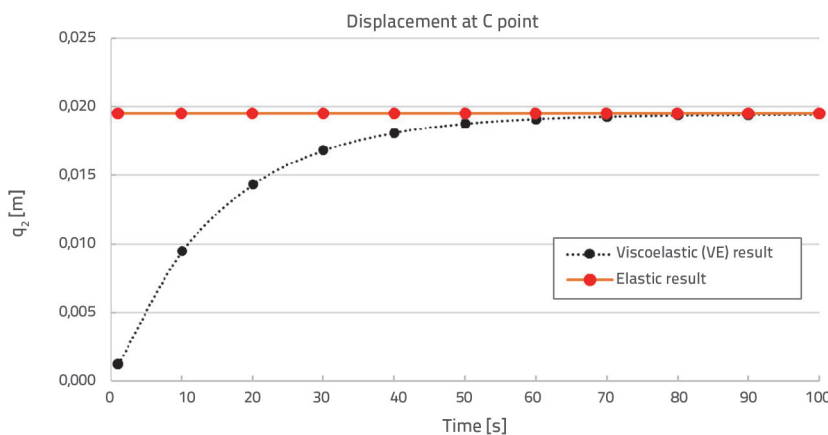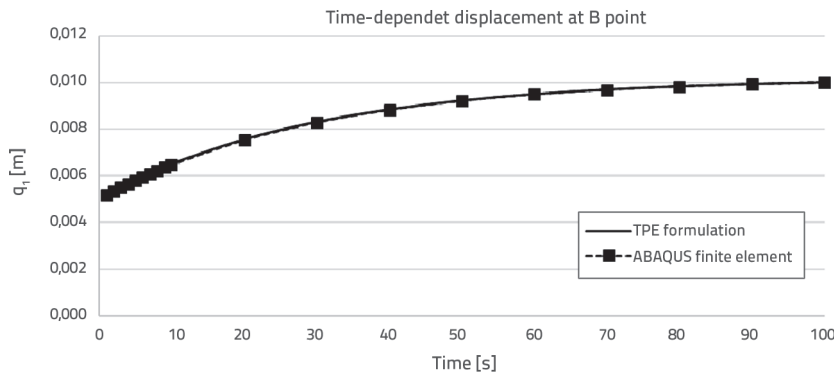Figure 7. Displacement (at point B) versus time plot for statically indeterminate elastic and VE cases

Figure 8. Displacement (at point C) versus time plot for statically indeterminate elastic and VE cases

Figure 9. Comparison of TPE formulation and ABAQUS result at B point

Figure 10. Comparison of TPE formulation and ABAQUS result at C point

viscoelastic bodies. The principle of the TPE is concerned with determining the function which makes a given functional stationary. The uniqueness of this study is that the energy functions of statically indeterminate VE bar-type structures in the Laplace domain were obtained for the first time. To convert the obtained results into the real domain, an inverse Laplace transform technique was applied. Three numerical examples were presented to ensure the accuracy and demonstrate the potential of the proposed procedure. Different mechanical analogues were used to represent the time-dependent behaviour of the viscoelastic materials. The proposed formulation has good potential to represent the time-dependent behaviour of a VE medium subjected to loading. It is worth noticing that the proposed method, which is theoretically simple to understand and easy to implement practically, proves efficient for building the results of any constitutive models of linear viscoelasticity. Rebuilding the set of equations to be solved depending on the type of VE material model and structural elements in a few straightforward steps is the primary advantage of the proposed method. This study presents highly accurate solutions for both statically determinate and indeterminate VE systems. To the best of our knowledge, this is the first study devoted to the analysis of statically indeterminate bar-type structures that exhibit VE characteristics using a systematic procedure based on the energy principle. The results presented here are novel. Beginning with a simple bar model, this study takes the author and her co-workers through increasingly complex problems of solid mechanics, such as VE structures, which consist of an assemblage of two or more bars and are subjected to bending, shear, and axial loads, called frames, and subject to axial forces, only called truss in 2D or 3D, and problems involving deformations, stability, and vibrations of VE bodies.

The SLS model represents the instantaneous strain when a load is applied suddenly. With regard to this discussion, it can be concluded that the proposed formulation has good potential to represent the time-dependent behaviour of the VE medium subjected to loading. It is worth noting that the proposed method, which is theoretically simple to understand and easy to implement, proves to be efficient for building the results of any constitutive model of linear viscoelasticity.

Rebuilding the set of equations to be solved depending on the type of VE material model and structural elements in a few straightforward steps is the primary advantage of the proposed method. The analysis of systems with different numbers of elements, viscoelastic material models, and loading types can be performed directly using a few simple steps.

6. Conclusion

Understanding the role of material behaviour in engineering problems, which are mathematical models of physical situations, is crucial. Several newly developed materials exhibit viscoelastic characteristics that are difficult to quantify. This study was performed to help researchers gain a better understanding of a new solution technique based on the minimum potential energy principle for analysing

Appendix

A transform of the first derivative:

$$L\left\{\frac{d}{dt}f\right\} = s\overline{f}(s) - f(0) \tag{A.1}$$

Transform of the convolution:

$$L(f * g) = \overline{f}(s) - \overline{g}(s) \tag{A.2}$$

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